

# Aharonov-Bohm conductance of a disordered single-channel quantum ring

P.M. Shmakov, A.P. Dmitriev and V.Yu.Kachorovskii

*A.F. Ioffe Physical-Technical Institute, 26 Polytechnicheskaya Street, St. Petersburg, 194021, Russia*

We study the effect of weak disorder on tunneling conductance of a single-channel quantum ring threaded by magnetic flux. We assume that temperature is higher than the level spacing in the ring and smaller than the Fermi energy. In the absence of disorder, the conductance shows sharp dips (antiresonances) as a function of magnetic flux. We discuss different types of disorder and find that the short-range disorder broadens antiresonances, while the long-range one leads to arising of additional resonant dips. We demonstrate that the resonant dips have essentially non-Lorentzian shape. The results are generalized to account for the spin-orbit interaction which leads to splitting of the disorder-broadened resonant dips, and consequently to coexisting of two types of oscillations (both having the form of sharp dips): Aharonov-Bohm oscillations with magnetic flux and Aharonov-Casher oscillations with the strength of the spin-orbit coupling. We also discuss the effect of the Zeeman coupling.

PACS numbers:

## Introduction

The Aharonov-Bohm (AB) effect<sup>1</sup> is one of the beautiful manifestations of the wave nature of electrons. The key physical issue — the sensitivity of the phase of an electronic wavefunction to a magnetic flux — enables the design of quantum AB interferometers<sup>2–20</sup> that can be tuned by an external magnetic field. Such interferometers occupy a worthy place in the quantum interferometry based on low-dimensional electronic nanosystems. A *single-channel ballistic* ring tunnel-coupled to the leads and threaded by the magnetic flux is the simplest realization of the AB interferometer (see Fig. 1). The interference of clockwise and counterclockwise electron trajectories manifests itself in the oscillations of the ring conductance  $G(\phi)$  with the period 1 (here  $\phi$  is the magnetic flux measured in the units of the flux quantum  $hc/e$ ).<sup>1,2</sup>

At low temperature  $T$  and weak tunneling coupling, AB conductance exhibits narrow resonant peaks both in clean and disordered single-channel rings<sup>21</sup> (see also Refs. 22–26 for discussion of disordered case). The peak arises each time when one of the field-dependent energy levels in the ring crosses the Fermi energy  $E_F$ . Hence, the positions of the AB resonances depend on  $E_F$ <sup>21</sup> (AB resonances are also affected by the Coulomb blockade<sup>27,28</sup>). Based on this physical picture one could expect the suppression of the resonance structure at  $T \gg \Delta$ , where  $\Delta$  is the level spacing in the ring. Remarkably, this naive expectation is incorrect and the interference effects are not entirely suppressed by the thermal averaging. Specifically, for  $T \gg \Delta$  the conductance of the noninteracting ring with weak tunnel coupling to the contacts exhibits sharp narrow dips (antiresonances) at  $\phi = 1/2 + n$ , where  $n$  is an arbitrary integer number (see Fig. 2).<sup>29,30</sup> It was also shown that the electron-electron interaction leads to arising of a fine structure of the antiresonances: each antiresonance splits into a series of narrow dips which correspond to blocking of the tunneling current by the persistent one<sup>30</sup> (in contrast to the Coulomb blockade

this effect is robust to increasing of temperature).

Additional physics comes into play in the presence of the spin-orbit (SO) interaction. In particular, the rotation of the electron spin in the built-in SO magnetic field results in a spin phase shift between clockwise and counterclockwise waves. This phase is additional with respect to AB phase and exists even at zero external magnetic field ( $\phi = 0$ ) so that zero-field conductance exhibits the Aharonov-Casher (AC) effect:<sup>31,32</sup> periodic oscillations with the strength of the SO coupling. The AC oscillations were the focus of intensive theoretical<sup>33–49</sup> research and their signatures were observed experimentally.<sup>50,51</sup> Recently, we demonstrated that these oscillations are also not suppressed by thermal averaging,<sup>52</sup> in a full analogy with the AB ones. Specifically, at  $T \gg \Delta$ , SO interaction splits AB antiresonances into pairs of symmetrical (with respect to  $\phi = 1/2 + n$ ) antiresonances. We also showed that the Zeeman interaction leads to appearance of two additional negative peaks on each period<sup>52</sup>.

What, to the best of our knowledge, has not been discussed in the literature is the effect of *disorder* on the tunneling conductance through a *single-channel* ring at relatively high temperatures,  $T \gg \Delta$ .<sup>53</sup> The aim of the current study is to fill this gap.

In this paper, we study the tunneling transport of non-interacting electrons through a disordered single-channel quantum ring of length  $L$  threaded by a magnetic flux  $\phi$ . We assume that  $T$  is much smaller than the Fermi energy  $E_F$  but large compared to  $\Delta = 2\pi\hbar v_F/L$  (throughout the paper we linearize electron spectrum near  $E_F$ , thus neglecting small variation of the electron velocity within the temperature band). The tunneling coupling characterized by tunneling probability  $\gamma$  is assumed to be weak,  $\gamma \ll 1$ , which implies that the ring is almost closed.

We discuss different types of disorder and find that the short-range disorder broadens antiresonances at  $\phi = n + 1/2$  while the long-range one leads to arising of additional antiresonances at  $\phi = n$ . We also find that the resonant dips have essentially non-Lorentzian shape. The results are generalized to account for the spin-orbit interaction

which leads to splitting of the disorder-broadened resonant dips, and consequently to coexisting of two types of oscillations (both having the form of sharp periodic dips): Aharonov-Bohm oscillations with magnetic flux and Aharonov-Casher oscillations with the strength of the spin-orbit coupling. Additional disorder-broadened resonant dips arise in the presence of the Zeeman coupling.

## I. CLEAN RING

We start with discussion of the high-temperature conductance of the clean ring following Refs. 30,52. This section aims to introduce basic notions and clarify our approach to the problem. Later this approach will be generalized to describe the effect of disorder.

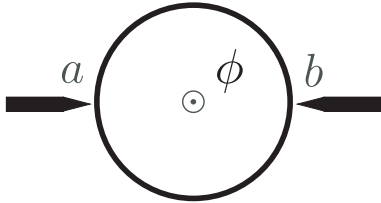


FIG. 1: The ring threaded by magnetic flux  $\phi$ .

The conductance is given by the Landauer formula:

$$G(\phi) = \frac{e^2}{\pi h} \mathcal{T}(\phi), \quad (1)$$

where

$$\mathcal{T}(\phi) = \langle \mathcal{T}(\phi, E) \rangle_E = - \int \mathcal{T}(\phi, E) \frac{\partial f}{\partial E} dE, \quad (2)$$

is the thermal average of the transmission coefficient  $\mathcal{T}(\phi, E)$  and  $f(E)$  is the Fermi-Dirac function (here we take into account double spin degeneracy).

We consider symmetrical setup (see Fig. 1) with identical point contacts described by the scattering matrix,

$$S = \begin{bmatrix} t_r & t_{out} & t_{out} \\ t & t_b & t_{in} \\ t & t_{in} & t_b \end{bmatrix}, \quad (3)$$

whose elements<sup>21</sup>

$$\begin{aligned} t_{in} &= \frac{1}{1+\gamma}, & t_b &= -\frac{\gamma}{1+\gamma}, \\ t = t_{out} &= \frac{\sqrt{2\gamma}}{1+\gamma}, & t_r &= -\frac{1-\gamma}{1+\gamma}. \end{aligned} \quad (4)$$

represent amplitudes of scattering from three incoming channels (1, 2, 3) to three outgoing ones (1', 2', 3') (see Fig. 3). Here  $\gamma$  is a real parameter characterizing the

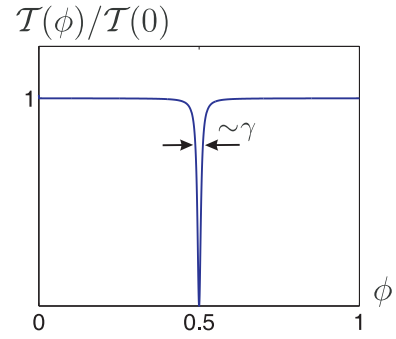


FIG. 2: Antiresonance in high-temperature transmission coefficient in the absence of disorder.

strength of the tunneling coupling to the contact: weak coupling corresponds to  $\gamma \ll 1$ , while an open contact is described by  $\gamma \sim 1$ .

The transmission amplitude can be calculated by summation of the amplitudes of all the trajectories connecting contact  $a$  and contact  $b$ , including the trajectories with backscatterings by contacts (the processes  $2 \rightarrow 2'$  and  $3 \rightarrow 3'$  on Fig. 3). Let us denote by  $n$  the number of times the electron passes the contact  $b$ , without exiting the ring ( $n = 0, 1, \dots$ ). The trajectories with a given  $n$  consist of the odd number  $2n + 1$  of semicircles and thus have the same length  $L_n = L(n + 1/2)$ . The sum of the amplitudes of such trajectories can be written as  $\beta_n \exp(ikL_n)$ , where  $k = \sqrt{2mE}/\hbar$  is the electron wavenumber. Hence, the transmission amplitude is written as

$$t(\phi, E) = \sum_{n=0}^{\infty} \beta_n \exp(ikL_n), \quad (5)$$

Next, we separate contributions of trajectories ending with lower and upper semicircle thus writing  $\beta_n = \beta_n^+ + \beta_n^-$ . Introducing vector  $\beta_n$  with two components,  $\beta_n^+$  and  $\beta_n^-$ , one may easily derive the following recurrence relations:

$$\beta_{n+1} = \hat{A} \beta_n, \quad (6)$$

where the the matrix  $\hat{A}$  is given by

$$\hat{A} = \begin{bmatrix} t_{in}^2 e^{-2\pi i \phi} + t_b^2 & t_b t_{in} (e^{-2\pi i \phi} + 1) \\ t_b t_{in} (e^{2\pi i \phi} + 1) & t_{in}^2 e^{2\pi i \phi} + t_b^2 \end{bmatrix} \quad (7)$$

$$= \frac{1}{(1+\gamma)^2} \begin{bmatrix} e^{-2\pi i \phi} + \gamma^2 & -\gamma(e^{-2\pi i \phi} + 1) \\ -\gamma(e^{2\pi i \phi} + 1) & e^{2\pi i \phi} + \gamma^2 \end{bmatrix}, \quad (8)$$

The element  $A_{ij}$  [multiplied by  $\exp(ikL)$ ] is the sum of the amplitudes of the trajectories starting at the contact  $b$  and making a single return to the same contact (indices  $i$  and  $j$  specify, respectively, the final and initial directions of motion:  $i = \pm, j = \pm$ ). The components of the vector  $\beta_0$ ,

$$\beta_0^+ = t t_{out} e^{-i\pi \phi}, \quad \beta_0^- = t t_{out} e^{i\pi \phi}, \quad (9)$$

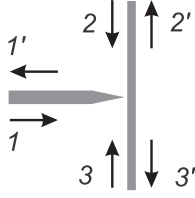


FIG. 3: Scattering on contacts: the amplitude  $t$  corresponds to processes  $1 \rightarrow 2'$  and  $1 \rightarrow 3'$ ,  $t_{out}$  - to  $2 \rightarrow 1'$  and  $3 \rightarrow 1'$ ,  $t_r$  - to  $1 \rightarrow 1'$ ,  $t_{in}$  - to  $2 \rightarrow 3'$  and  $3 \rightarrow 2'$ ,  $t_b$  - to  $2 \rightarrow 2'$  and  $3 \rightarrow 3'$ .

yield contributions of shortest counterclockwise and clockwise trajectories, respectively.

Using Eq. (5), we express transmission coefficient in terms of  $\beta_n$ :

$$\mathcal{T}(\phi, E) = |t(\phi, E)|^2 = \sum_{n,m=0}^{\infty} \beta_n \beta_m^* e^{ik(L_n - L_m)}. \quad (10)$$

The terms with  $n \neq m$  in Eq. (10) vanish after thermal averaging in the discussed case  $T \gg \Delta$ , so that the expression for the averaged transmission coefficient becomes

$$\mathcal{T}(\phi) = \sum_{n=0}^{\infty} |\beta_n|^2 = \sum_{n=0}^{\infty} \left| (\mathbf{e}, \hat{A}^n \beta_0) \right|^2. \quad (11)$$

where vector  $\mathbf{e}$  has components (1,1). The calculation of the sum entering Eq.(11) is quite cumbersome but straightforward (see Appendix A). Using Eq. (A5) we obtain:<sup>30</sup>

$$\mathcal{T}(\phi) = \frac{2\gamma \cos^2 \pi\phi}{\gamma^2 + \cos^2 \pi\phi}. \quad (12)$$

The dependence  $\mathcal{T}(\phi)$  is shown on Fig. 2. The physical explanation of the dip (antiresonance) at  $\phi = 1/2$  is quite simple (here and below we consider the interval  $0 < \phi < 1$ ). Let us demonstrate that at  $\phi = 1/2$  the contribution of any trajectory is exactly canceled by contribution of the trajectory mirrored with respect to the line connecting  $a$  and  $b$ . Indeed, the sum of the amplitudes of these two trajectories is proportional to  $e^{ikL_n} (e^{i(2|m|+1)\pi\phi} + e^{-i(2|m|+1)\pi\phi})$ , where  $m$  is a difference between the number of clockwise and counterclockwise revolutions,  $|m| \leq n$ . At  $\phi = 1/2$  this sum turns to zero for any  $k$ . Thus, the antiresonance is due to the destructive interference of mirrored paths.

For weak tunneling coupling,  $\gamma \ll 1$ , the antiresonance is well approximated by the Lorentz-shape dip:

$$\mathcal{T}(\phi) \approx 2\gamma \frac{\pi^2 \delta\phi^2}{\gamma^2 + \pi^2 \delta\phi^2}, \quad (13)$$

where  $\delta\phi = \phi - 1/2$ .

It is worth noting, that in the vicinity of antiresonance one can neglect the backscattering on the contacts. Indeed, at  $|\phi - 1/2| \sim \gamma$  the off-diagonal elements of  $\hat{A}$

which are proportional to  $t_b \sim \gamma$  multiplied by small factors  $1 + \exp(\pm i2\pi\phi) \sim \gamma$  which implies that backscattering is effectively suppressed by a factor  $\gamma$ . Physically, the effective suppression of backscattering is explained by destructive interference of two processes. In the first process an electron is reflected by a contact (say, contact  $b$ ) and returns to this contact after one revolution around the ring. The amplitude of such a process is  $t_b t_{in} \exp(\pm i2\pi\phi)$  (the sign is prescribed by direction of the propagation) where the amplitude  $t_{in}$  appeared because the contact  $a$  was passed without reflection. In the second process the electron passes the contact  $b$  without reflection and then is reflected by contact  $a$  and returns to  $b$ . The corresponding amplitude is given by  $t_{in} t_b$ . Evidently, for  $\phi = 1/2$  the amplitudes of these processes exactly cancel each other.

It is worth noting that backscattering is important in vicinity of integer values of flux.<sup>30</sup> In particular, putting  $t_b = 0$  in Eq. (7) we come to incorrect conclusion that there are resonant peaks at  $\phi = n$  in the evident contradiction with Eq. (12).

The approach discussed above allows one to find transmission coefficient for arbitrary  $\gamma$  and  $\phi$ . However, it is technically cumbersome and lacks physical transparency. Below we derive the main result of this section, Eq. (13), by using an alternative method. This method is valid only in the vicinity of  $\phi = 1/2$ , where backscattering by contacts can be neglected. However, it has a number of advantages compared to the first one: it is more illustrative physically, and much more easily generalized to account for disorder.

The key idea is that for  $\gamma \ll 1$  and  $\delta\phi \ll 1$  the tunneling amplitude through the ring may be presented as a sum of the transition amplitudes through intermediate states corresponding to quasistationary levels of almost closed ring. The appropriate analytical expression is derived in Appendix B and reads

$$\mathcal{T} \approx \hbar^2 v_F^2 t^2 t_{out}^2 \left\langle \left| G_{E+i\Gamma/2}(0, L/2) \right|^2 \right\rangle_E, \quad (14)$$

where

$$G_{E+i\Gamma/2}(0, L/2) = \frac{1}{\hbar v_F} \sum_l C_l(E) = \sum_l \frac{\psi_l^*(0) \psi_l(L/2)}{E - \epsilon_l + i\Gamma/2}, \quad (15)$$

$G_E(0, L/2)$  is the Green function of the closed ring, describing the transition from the contact  $a$  to contact  $b$ ,  $E$  is the energy of the tunneling electron, and  $\epsilon_l$  are the electron energies in the closed ring, corresponding to wave functions  $\psi_l(x)$ . The quantities

$$C_l(E) = \hbar v_F \frac{\psi_l^*(0) \psi_l(L/2)}{E - \epsilon_l + i\Gamma/2} \quad (16)$$

are the amplitudes of transition through the corresponding quasistationary states (see Fig. 4). Here  $\Gamma$  is the tunneling rate given by

$$\Gamma = \frac{2\Delta\gamma}{\pi}. \quad (17)$$

In the closed clean ring there are two types of the electron states, corresponding to counterclockwise and clockwise propagation, labeled below by indices  $l = (n, +)$  and  $l = (n, -)$  respectively. Wave functions and energies of these states read

$$\psi_n^\pm(x) = \frac{e^{\pm i 2\pi n x / L}}{\sqrt{L}}, \quad \epsilon_n^\pm(\phi) = \Delta(n \pm \phi) + c, \quad (18)$$

where constant  $c = E_F - \Delta n_F$  arises in course of linearization of the spectrum near the Fermi energy [here  $n_F$  obeys the following equation:  $E_F = 2\hbar^2\pi^2 n_F^2 / mL^2$ ]. As seen, for  $\phi = 1/2$  each level is double-degenerate,  $\epsilon_n^+(1/2) = \epsilon_{n+1}^-(1/2)$ . Finite  $\delta\phi$  lifts the degeneracy of these levels:

$$\epsilon_n^+ = E_n + \Delta\delta\phi, \quad \epsilon_{n+1}^- = E_n - \Delta\delta\phi, \quad (19)$$

where  $E_n = \Delta(n + 1/2) + c$ .

Let us now demonstrate that Eqs. (14), and (15) yield  $\mathcal{T}(1/2) = 0$ . Indeed, from Eqs. (16), and (18) we easily find:

$$C_n^+ = -C_{n+1}^-, \quad \text{for } \phi = 1/2, \quad (20)$$

so that  $G_E(0, L/2)|_{\phi=1/2} \equiv 0$  for any  $E$  and transmission coefficient turns to zero even before energy averaging. The minus sign in the r.h.s. of Eq. (20) appeared due to the property

$$\psi_n^+(L/2) = -\psi_{n+1}^-(L/2). \quad (21)$$

This is the property that leads to destructive interference and formation of the dip in the tunneling conductance. Physically, this is an alternative way to describe a compensation of mirrored paths discussed above.

Next we derive Eq. (13). First, we rewrite (14) as follows

$$\mathcal{T} = -t^2 t_{out}^2 \sum_{l, l'} C_l^*(E) C_{l'}(E) \frac{\partial f}{\partial E} dE. \quad (22)$$

The double sum in this equation contains “classical” terms proportional to  $|C_l|^2$ , as well as the interference ones,  $C_l^* C_{l'}$  (with  $l \neq l'$ ). The main contribution to the integral in Eq. (22) comes from the vicinities of poles (with the size on the order of  $\Gamma$ ) of the amplitudes  $C_l^*(E)$  and  $C_{l'}(E)$ . It is easy to see from Eqs. (16) and (22) that interference terms are comparable with classical ones only if  $|\epsilon_l - \epsilon_{l'}| \lesssim \Gamma$ . For  $\delta\phi \ll 1$ , energies  $\epsilon_n^+$  and  $\epsilon_{n+1}^-$  are close to each other,  $\epsilon_n^+ - \epsilon_{n+1}^- \approx 2\Delta\delta\phi$  (see Fig. 4), and differ from the energies of other levels by a much larger distance ( $\Delta$  or larger). The interference contributions to Eq. (22) containing products of the amplitudes from different pairs, can be neglected compared to the “classical” terms and to the interference terms, containing products of the amplitudes from the same pair. Therefore, we can rewrite Eq. (22) as a sum over pairs of close levels

$$\mathcal{T} \approx -t^2 t_{out}^2 \sum_n \int |C_n^+ + C_{n+1}^-|^2 \frac{\partial f}{\partial E} dE. \quad (23)$$

Hence, the paths over which electron passes through the ring can be split into pairs of interfering paths, corresponding to quasi-degenerate intermediate states  $\psi_n^+$  and  $\psi_{n+1}^-$  (see Fig. 4).

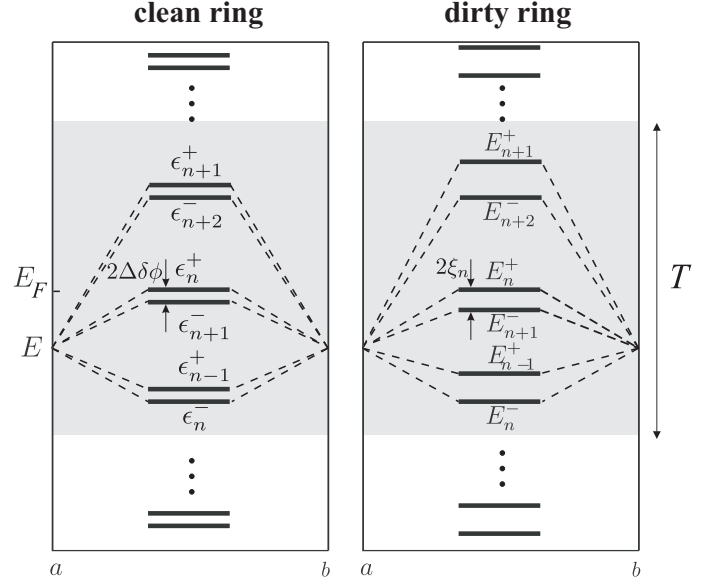


FIG. 4: Tunneling of an electron through pairs of close levels in the ring. For a clean ring the distance between levels in all pairs is the same and is given by  $2\Delta\delta\phi$ . For a ring with disorder this distance increases due to the repulsion of the levels in the disorder potential and becomes  $n$ -dependent:  $E_n^+ - E_{n+1}^- = 2\xi_n = 2\sqrt{\Delta^2\delta\phi^2 + |V_n|^2}$ .

Now we demonstrate that contributions of different pairs in Eq. (23) in fact differ by the thermal factor only, so that the problem can be reduced to analysis of the transition through a single pair. First we notice, that the energy dependence of the thermal factor  $\partial f / \partial E$  is smooth and in the  $n$ -th term of the sum one may replace  $\partial f / \partial E$  with  $(\partial f / \partial E)|_{E=E_n}$ . Next we change the integration variable in this term:  $E \rightarrow \epsilon = E - E_n$ . Then, dependence on  $n$  remains only in the factors  $\psi_n^{\pm*}(0)\psi_n^\pm(L/2) = (-1)^n / L$ . This dependence disappears after calculation of modulus squared in Eq. (23). Finally, we calculate sum over  $n$ ,  $\sum_n (-\partial f / \partial E)|_{E=E_n} \approx 1/\Delta$  (here we use inequality  $T \gg \Delta$ ), and arrive to the following equation

$$\mathcal{T} = \frac{t^2 t_{out}^2 \Delta}{4\pi^2} \int_{-\infty}^{\infty} d\epsilon \left| \frac{1}{\epsilon - \epsilon^+ + i\Gamma/2} - \frac{1}{\epsilon - \epsilon^- + i\Gamma/2} \right|^2 \quad (24)$$

which expresses  $\mathcal{T}$  in terms of transition through a single pair of close levels with energies  $\epsilon^\pm = \pm\Delta\delta\phi$ . The minus sign in front of the second fraction in Eq. (24) appeared due to the property (21). One may separate in Eq. (24) “classical” contribution,  $\mathcal{T}_{cl}(\phi)$  (integral from the sum of the squared amplitudes), from the interference one,  $\mathcal{T}_{int}(\phi)$ . Performing integration, we find  $\mathcal{T}_{cl}(\phi) = 2\gamma$  and  $\mathcal{T}_{int}(\phi) = -2\gamma^3/(\gamma^2 + \pi^2\delta\phi^2)$ . Summing these terms we restore Eq. (13). We notice that, as expected, interfer-

ence term gives significant contribution only in the region of small  $\delta\phi$ :  $|\delta\phi| \lesssim \gamma$ .

## II. THE RING WITH IMPURITIES

In the above calculations we considered the case of the clean ring. Now we discuss the effect of disorder on the high-temperature conductance of the ring.

### A. Long-range disorder

One of the realizations of the disorder is a weak smooth random potential with the correlation length much exceeding the electron Fermi wavelength. In this case, backscattering by disorder is exponentially suppressed, so that the potential only leads to the additional phase shift between the right and left-moving electron waves propagating from contact  $a$  to contact  $b$  along upper and lower shoulder of interferometer, respectively (with zero winding number). We denote the disorder-induced phase difference between these two waves as  $\Psi(E)$ . Such an interferometer is evidently equivalent to the clean one having two arms with the lengths  $(L - a)/2$  and  $(L + a)/2$ , where  $a \approx \Psi(E_F)/k_F$ . The conductance of the latter interferometer was calculated in Ref. 30. From Eq. (A3) of Ref. 30 we find

$$\mathcal{T}(\phi) = F[\sin(\pi\phi), \sin(\Psi/2)] + F[\cos(\pi\phi), \cos(\Psi/2)], \quad (25)$$

where

$$F(x, y) = 2\gamma \frac{x^2 y^2}{x^2 + \gamma^2 y^2}. \quad (26)$$

This equation is valid provided that  $T(d\Psi/dE)_{E=E_F} \ll 1$ . As seen, for  $\Psi \neq 0$  there are two dips in the conductance (at  $\phi = 1/2$  and at  $\phi = 0$ ), the widths and the depths of the dips being oscillating functions of  $\Psi = \Psi(E_F)$  [in particular,  $\mathcal{T}(0) = 2\gamma \cos^2(\Psi/2)$ ,  $\mathcal{T}(1/2) = 2\gamma \sin^2(\Psi/2)$ ]. Hence, long-range disorder leads to appearance of the additional antiresonance in the conductance at  $\phi = 0$  and modifies the antiresonance near  $\phi = 1/2$ .

### B. Short-range disorder

#### 1. Calculation of the transmission coefficient

Another realization of disorder is the potential created by weak short-range impurities, randomly distributed along the ring with the concentration  $n_i$ . Let us characterize the strength of disorder by the scattering rate in the infinite wire calculated by the golden rule. For short-range potential, transport and quantum scattering rates coincide and are given by  $1/\tau = 2|r|^2 v_F n_i$ , where  $r$  is

the reflection amplitude for a single impurity ( $|r| \ll 1$ ). Substituting in this equation  $n_i = N/L$  (here  $N$  is the number of impurities in the ring) we get

$$\frac{1}{\tau} = \frac{N|r|^2 \Delta}{\pi \hbar}. \quad (27)$$

We restrict ourselves to discussion of the ballistic case,<sup>54</sup>  $v_F \tau \gg L$ , or, equivalently,  $N|r|^2 \ll 1$ .

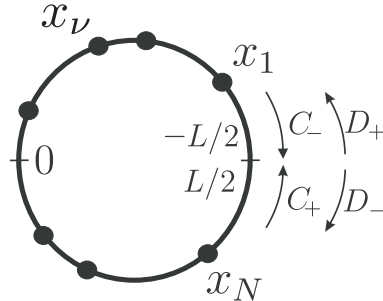


FIG. 5: A ring with impurities.

We will see that the main effect of the short-range potential is the broadening of the antiresonances. One could expect that scattering by disorder leads to essential increase of the resonance width, when  $\tau$  becomes shorter than lifetime of the electron in the ring,  $\hbar/\Gamma$ , which implies  $N|r|^2 \gg \gamma$ . Another expectation is that in the regime  $N|r|^2 \gg \gamma$ , when electron experiences many scatterings during the lifetime and therefore acquires random phase, the interference is suppressed, and, consequently, the depth of the dip essentially decreases. However, we will show that the scattering on the impurities comes into play at much smaller disorder strength when  $N|r|^2 \sim \gamma^2$ , so that for  $\gamma^2 \ll N|r|^2 \ll 1$  the dip is essentially broadened. Also, in contrast to the naive expectation, its depth remains on the order of  $\gamma$ .

Now we generalize the method, introduced in the first part of the previous section. To do this we should modify the matrix  $\hat{A}$ , taking into account the scattering on the impurities. This matrix becomes complicated, since it includes the amplitudes of all the trajectories with scatterings on both contacts and impurities, after which an electron returns to contact  $b$ . However, in the case  $\delta\phi \ll 1$ ,  $\gamma \ll 1$  and  $N|r|^2 \ll 1$  the matrix  $\hat{A}$  can be simplified.

As a first step we expand matrix  $\hat{A}$  in a Taylor series up to the first order with respect to  $\gamma$ , denoting  $\hat{A}_0 = \hat{A}|_{\gamma=0}$ .

For the clean ring ( $r = 0$ ) we have

$$\hat{A} \approx (1 - 2\gamma)\hat{A}_0 \approx (1 - 2\gamma) \begin{bmatrix} e^{-2\pi i \phi} & 0 \\ 0 & e^{2\pi i \phi} \end{bmatrix}. \quad (28)$$

We neglected off-diagonal elements of  $\hat{A}$ , since, as we explained above, backscattering on the contacts is effectively suppressed at  $\delta\phi \ll 1$ . Now we write the expansion for dirty ring in a way that reproduces Eq. (28) for  $r = 0$

$$\hat{A} = \hat{A}_0 - 2\gamma(\hat{A}_0 + \delta\hat{A}) + \dots \quad (29)$$



Here both  $\hat{A}_0$  and  $\delta\hat{A}$  depend on  $r$ . The matrix  $\delta\hat{A}$  should vanish at  $r = 0$ , so that  $\delta\hat{A} \propto r$  at small  $r$ . Calculations using Eq. (A5) show that one can neglect the term  $\gamma\delta\hat{A}$  in Eq. (29) (as well as terms on the order of  $\gamma^2$  and higher). Physically, this implies neglect of the processes involving both scattering by impurities and forward scattering by contacts during one revolution around the ring.

Let us now discuss the properties of the matrix  $\hat{A}_0$ . According to the definition of the matrix  $\hat{A}$  (see previous section) the matrix  $e^{ikL}\hat{A}_0$  relates the amplitudes  $C_{\pm}$  and  $D_{\pm}$  of the incoming and outgoing waves, respectively, at the point  $b$  (see Fig. 5)

$$\begin{bmatrix} C_+ \\ C_- \end{bmatrix} = e^{ikL}\hat{A}_0 \begin{bmatrix} D_+ \\ D_- \end{bmatrix} \quad (30)$$

Hence, the matrix  $e^{ikL}\hat{A}_0$  is the  $S$ -matrix describing a complex scatterer consisting of  $N$  impurities, located at points  $x_{\nu}$  between  $x = -L/2$  and  $x = L/2$ . Having in mind that this matrix should be unitary and taking into account the time reversal symmetry we write this matrix in the most general form:

$$\hat{A}_0 = e^{i\alpha} \begin{bmatrix} \sqrt{1-|R|^2}e^{-2i\pi\phi} & R \\ -R^* & \sqrt{1-|R|^2}e^{2i\pi\phi} \end{bmatrix}. \quad (31)$$

Here  $\alpha$  is the small forward scattering phase for a complex scatterer consisting of  $N$  impurities. This phase is added to the geometrical phase  $kL$  and, therefore, drops out after thermal averaging. The off-diagonal element,  $R$ , is, up to a phase factor, the reflection amplitude from a complex of  $N$  impurities. One can expand  $R$  with respect to  $r$ . In the lowest order in  $r$  we obtain

$$R \approx r \sum_{\nu=1}^N e^{-2ikx_{\nu}}. \quad (32)$$

This expression takes into account only one backscattering on impurities during a revolution around the ring, and is valid in the case  $N|r|^2 \ll 1$ .

As we see, the matrix  $\hat{A}$  is now dependent on  $k$ , so that Eq.(11) does not generally follow from Eq.(10). However, if we assume that impurities are randomly distributed along the ring (some special non-random distributions will be discussed at the end of Sec. IIB) the terms with  $n \neq m$  in Eq.(10) do not survive the averaging over  $k$ . Therefore we can use Eq.(11) while performing averaging over  $k$ :

$$\mathcal{T} = \left\langle \sum_{n=0}^{\infty} \left| (\mathbf{e}, \hat{A}^n \boldsymbol{\beta}_0) \right|^2 \right\rangle_k. \quad (33)$$

Since the probability to scatter on an impurity before the first visit of the point  $b$  is small, we can neglect the scattering terms in the vector  $\boldsymbol{\beta}_0$  entering Eq. (33) and use Eq. (9). The sum in Eq. (33) can be calculated with the use of Eq. (A5) which simplifies after expansion of the numerator and denominator with respect to  $\gamma, r$  and

$\delta\phi$ . Calculations yield the following expression for the transmission coefficient:

$$\mathcal{T} \approx 2\gamma \left\langle \frac{\pi^2\delta\phi^2 + |r|^2(\sum_{\nu} \sin 2kx_{\nu})^2/4}{\pi^2\delta\phi^2 + \gamma^2 + |r|^2 \sum_{\nu} e^{2ikx_{\nu}}|^2/4} \right\rangle_k. \quad (34)$$

This equation is valid provided that  $\gamma \ll 1, N|r|^2 \ll 1$  and  $\delta\phi \ll 1$ . The relation between  $\sqrt{N}|r|$  and  $\gamma$  can be arbitrary.

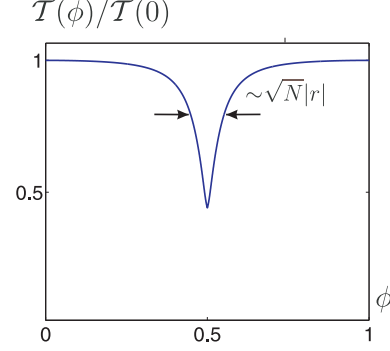


FIG. 6: Antiresonance in high-temperature transmission coefficient in the presence of  $N$  randomly distributed impurities for  $\sqrt{N}|r| \gg \gamma$ .

The approach, discussed above, can be also used to calculate the transmission coefficient for arbitrary  $\phi$ . In particular, one can show that at  $\phi = 0$  there appears a small dip with the amplitude on the order of  $N|r|^2\gamma \ll \gamma$  (to obtain this result one should take into account backscattering by contacts). In the following discussion we focus only on the antiresonance at  $\phi = 1/2$ .

Let us first consider a ring with a single impurity. As seen from Eq. (34), the transmission coefficient is given by the Lorentz-shape antiresonance:

$$\mathcal{T} \approx 2\gamma \frac{\pi^2\delta\phi^2 + |r|^2/8}{\pi^2\delta\phi^2 + \gamma^2 + |r|^2/4}. \quad (35)$$

We see that the transmission coefficient at  $\phi = 1/2$  is no longer equal to zero and the antiresonance broadens so that its width becomes  $\sqrt{\gamma^2 + |r|^2/4}$ . We also find that the depth of the dip changes from  $2\gamma$  to  $\gamma$  with increasing  $|r|$ . In other words, in contrast to the antiresonance width, its depth remains the same order of magnitude.

In order to perform the averaging over  $k$  in the case of many impurities, we notice, that for random impurity distribution the averaging over  $k$  is equivalent to averaging over  $x_{\nu}$ :  $\langle \dots \rangle_k = \langle \dots \rangle_{x_1 \dots x_N}$ . For two impurities the average is easily calculated. The result reads

$$\mathcal{T} \approx \gamma \left( \frac{\pi^2\delta\phi^2 - \gamma^2}{\sqrt{(\pi^2\delta\phi^2 + \gamma^2)(\pi^2\delta\phi^2 + \gamma^2 + |r|^2)}} + 1 \right). \quad (36)$$

For the case  $N > 2$ , we rewrite Eq. (34) using the identities  $x^{-1} \equiv \int_0^{\infty} \exp(-tx)dt$ ,  $\exp(-x^2) \equiv \int_{-\infty}^{\infty} \exp(-y^2 +$

$2ixy)dy/\sqrt{\pi}$ , and get the following expression:

$$\mathcal{T} \approx 2\gamma \int_0^\infty \frac{dt}{4\pi t} e^{-4(\pi^2\delta\phi^2 + \gamma^2)t/|r|^2} \quad (37)$$

$$\times \int_{-\infty}^\infty d\xi d\eta e^{-\frac{\xi^2 + \eta^2}{4t}} \left( \frac{4\pi^2\delta\phi^2}{|r|^2} - \frac{\partial^2}{\partial \xi^2} \right) J_0^N(\sqrt{\eta^2 + \xi^2}),$$

or

$$\mathcal{T} \approx 2\gamma \int_0^\infty d\rho K_0 \left( \frac{2\rho\sqrt{\pi^2\delta\phi^2 + \gamma^2}}{|r|} \right)$$

$$\times \left( \frac{4\pi^2\delta\phi^2\rho}{|r|^2} - \frac{\partial}{2\partial\rho} \rho \frac{\partial}{\partial\rho} \right) J_0^N(\rho), \quad (38)$$

where  $K_0$  is the modified Bessel function of the second kind.

Assuming now that the number of impurities is large,  $N \gg 1$ , we get  $J_0^N(x) \approx \exp(-Nx^2/4)$  and after simple calculation obtain

$$\mathcal{T} \approx \frac{2\gamma}{s^2} \int_0^\infty dx \frac{\pi^2\delta\phi^2(1+x) + s^2/2}{(1+x)^2} \exp\left(-x \frac{\pi^2\delta\phi^2 + \gamma^2}{s^2}\right), \quad (39)$$

where  $s^2 = N|r|^2/4$ . This dependence is plotted in Fig. 6. We see that similar to the case of a single impurity, the transmission coefficient at  $\phi = 1/2$  is no longer equal to zero and the antiresonance broadens. It is also notable that for  $N \geq 2$  the dip has a non-Lorentzian shape.

Let us discuss two limiting cases. For  $\sqrt{N}|r| \ll \gamma$ , the minimal value of conductance is given by  $\mathcal{T}|_{\delta\phi=0} \approx N|r|^2/4\gamma$ , and the width of the antiresonances increases from  $\gamma$  to  $\gamma' = \gamma + \delta$  where

$$\delta \sim \frac{N|r|^2}{\gamma}. \quad (40)$$

The relative contribution of the disorder to the resonance width,  $\delta/\gamma \sim N|r|^2/\gamma^2 = \hbar/\gamma\Gamma\tau$ , is enhanced by a factor  $\gamma^{-1} \gg 1$  in comparison with naive expectation  $\hbar/\Gamma\tau$ . In the opposite limiting case,  $\sqrt{N}|r| \gg \gamma$ , we get  $\mathcal{T}|_{\delta\phi=0} \approx \gamma$ , so that the depth of the antiresonance is two times smaller compared with the case of clean ring, while the width is given by

$$\gamma' \sim \sqrt{N}|r|. \quad (41)$$

## 2. Two-level approximation

Next we discuss the obtained results in terms of transition amplitudes through intermediate quasi-stationary states of almost closed ring. We recall that in the vicinity of the flux  $\phi = 1/2$  the energy levels of the clean ring can be split into pairs of close levels corresponding to

clockwise and counterclockwise propagation of the electron inside the ring. Energy distance between levels in the pairs equals to  $2\Delta\delta\phi$  (see Fig. 4, left panel). For the case when impurity potential  $V(x) = \sum_{\nu=1}^N U(x - x_\nu)$  [here  $U(x - x_\nu)$  is the potential of the  $\nu$ -th impurity] is sufficiently weak (see corresponding criterion below) it can be simply accounted for by perturbation theory for two close levels.

Calculation yields for energies and wave functions of potential-disturbed states:

$$E_n^+ = E_n + \xi_n, \quad E_{n+1}^- = E_n - \xi_n, \quad (42)$$

$$\xi_n = \sqrt{\Delta^2\delta\phi^2 + |V_n|^2}, \quad (43)$$

$$\Psi_n^+ = \frac{\psi_n^+ + \psi_{n+1}^- V_n/W_n}{\sqrt{1 + |V_n|^2/W_n^2}}, \quad (44)$$

$$\Psi_{n+1}^- = \frac{\psi_{n+1}^- - \psi_n^+ V_n^*/W_n}{\sqrt{1 + |V_n|^2/W_n^2}}, \quad (45)$$

where  $W_n = \Delta\delta\phi + \xi_n$ , and

$$V_n = \int dx \psi_{n+1}^{*-}(x) V(x) \psi_n^+(x) = \frac{ir\Delta}{2\pi} \sum_{\nu} e^{2\pi i(2n+1)x_\nu/L}. \quad (46)$$

is the matrix element of the impurity potential expressed in terms of reflection amplitude  $r$ . For  $\delta\phi \ll 1$ , the inequality which ensures validity of the two-level perturbation theory is given by  $|V_n| \ll \Delta$ . For randomly distributed impurities, the amplitude of the potential is estimated as  $|V_n| \sim |r| \Delta \sqrt{N}$ . Hence, the perturbation theory applies for  $|r| \sqrt{N} \ll 1$ .

Let us first discuss the dependence of the conductance on  $\delta\phi$  qualitatively. First we note that due to the repulsion between levels in the impurity potential the minimal distance between levels in any pair is given by  $|V_n|$  (this distance corresponds to  $\delta\phi = 0$ ). For  $|V_n| \ll \Gamma$  similar to the case of a clean ring, there exist a dip in the transmission coefficient which arises due to the interference between transition amplitudes through pairs of close levels (see Fig. 4), while the "classical" term is featureless at  $\phi = 1/2$ .

In contrast, in the opposite limiting case,  $|V_n| \gg \Gamma$ , the energy distance between levels in any pair becomes much larger than  $\Gamma$ , and, consequently, contribution of the interference terms to the transmission coefficient is small compared to "classical" ones. This, however, does not lead to disappearance of the dip in the transmission coefficient. It turns out that in the dirty ring the "classical" terms acquire sharp dependence on  $\phi$  and decrease by a factor 2 within a narrow region  $\delta\phi \sim \sqrt{N}|r|$ .

Indeed, as seen from Eqs. (44)-(45), for  $|\Delta\delta\phi| \gg |V_n|$  the wave functions in the  $n$ -th pair are simply given by clockwise- and counterclockwise-moving waves  $\psi_n^+$  and  $\psi_{n+1}^-$ . The "classical" contribution to the transmission coefficient from each of these levels, say level  $(n, +)$ , is proportional to  $|\psi_n^+(0)|^2 |\psi_n^+(L/2)|^2 = 1/L^2$ . In the opposite case,  $|\Delta\delta\phi| \ll |V_n|$ , disorder potential strongly mixes clockwise- and counterclockwise-

propagating waves. Consider, for simplicity, the case  $\delta\phi = 0$ . From Eqs. (44)-(45) we see that  $\Psi_n^+ = (\psi_n^+ + e^{i\varphi_n}\psi_{n+1}^-)/\sqrt{2}$  and  $\Psi_{n+1}^- = (\psi_n^+ - e^{-i\varphi_n}\psi_{n+1}^-)/\sqrt{2}$ , where  $\exp(i\varphi_n) = V_n/|V_n|$ . Averaging  $|\Psi_n^+(0)|^2|\Psi_n^+(L/2)|^2$  over random phase  $\varphi_n$  and taking into account Eq. (21) we obtain twice smaller value  $1/2L^2$ . This implies existence of a dip of a width  $\delta\phi \sim |V_n|/\Delta \sim \sqrt{N}|r|$  in the transmission coefficient:  $\mathcal{T}$  decreases by a factor 2 within this width.

The transition from the interference to “classical” picture of formation of the dip can be illustrated by an example of the ring with a single impurity. The conductance of such a ring is given by Eq. (35). The “classical” and interference contributions to this expression read

$$\mathcal{T}_{cl}(\phi) = \frac{2\gamma(\pi^2\delta\phi^2 + |r|^2/8)}{\pi^2\delta\phi^2 + |r|^2/4}, \quad (47)$$

$$\mathcal{T}_{int}(\phi) = -\frac{2\gamma^3(\pi^2\delta\phi^2 + |r|^2/8)}{(\pi^2\delta\phi^2 + |r|^2/4)(\pi^2\delta\phi^2 + \gamma^2 + |r|^2/4)}. \quad (48)$$

As seen, the interference contribution leads to formation of the dip at  $\gamma \gg |r|$  and can be neglected for  $\gamma \ll |r|$ .

The rigorous calculations of the conductance can be performed in a way analogous to the case of the clean ring. The transmission coefficient is expressed via a Green function of the closed ring and can be approximately presented as a sum over pairs of interfering paths through intermediate states  $\Psi_n^+$  and  $\Psi_{n+1}^-$ . Equations (16) and (23) still hold with the following replacement:  $\psi_n^\pm \rightarrow \Psi_n^\pm$  and  $\epsilon_n^\pm \rightarrow E_n^\pm$ . In contrast to the case of the clean ring, the summands in Eq (23) are now different not only due to the thermal factor  $(\partial f/\partial E)|_{E=E_n}$  but also because of the strong dependence of  $V_n$  on  $n$ . However, for random distribution of impurities the summation over  $n$  within the temperature window is equivalent to averaging over the impurity positions. After such averaging the dependence on  $n$  remains only in the thermal factor and the problem is again reduced to the case of transition through a single pair of close levels. The transmission coefficient may be written as a sum of classical and interference terms

$$\mathcal{T}(\phi) = \frac{t_{out}^2}{\Delta} \int_{-\infty}^{\infty} d\epsilon (\rho_{cl} + \rho_{int}). \quad (49)$$

Here  $\rho_{cl} = \langle |C_n^+|^2 + |C_{n+1}^-|^2 \rangle_{x_\nu}$ ,  $\rho_{int} = 2\text{Re}\langle C_n^{+*} C_{n+1}^- \rangle_{x_\nu}$ , where  $C_n^+, C_{n+1}^-$  are now expressed via energy  $\epsilon$  in the following way:  $C_n^+ = \hbar v_F \Psi_n^{+*}(0) \Psi_n^+(L/2)/(\epsilon - \xi_n + i\Gamma/2)$ ,  $C_{n+1}^- = \hbar v_F \Psi_{n+1}^{*-}(0) \Psi_{n+1}^-(L/2)/(\epsilon + \xi_n + i\Gamma/2)$ . Using Eqs. (42)-(49) after some algebra we arrive to Eq. (34).

### 3. Special impurity distributions

Above we assumed that impurities are randomly distributed along the ring. If this is not the case, the results might be quite different. The reason is that there exist some impurity distributions for which the summation

over  $n$  within the temperature band is not equivalent to the averaging over the impurity positions, so that calculation should be performed more carefully. One may check that the method discussed in Sec. II B 2 reproduces Eq. (34) with the replacement  $k \rightarrow k_n = 2\pi(n + 1/2)/L$  and  $\langle \dots \rangle_k \rightarrow -\Delta \sum_n (\partial f/\partial E)_{k=k_n} \dots$ . Let us now give some examples of special impurity distributions for which Eqs. (36)-(39) are invalid. If the impurities are distributed symmetrically with respect to the line  $(a, b)$  connecting the contacts, then  $\sum_\nu \sin 2k_n x_\nu = 0$  and, consequently,  $\mathcal{T}(1/2) = 0$ . On the language of trajectories this can be explained by cancelation of the contributions of the mirrored paths just as in the case of the clean ring. The width of the dip is on the order of  $|r|\sqrt{N}$ . If impurities are distributed symmetrically with respect to the line perpendicular to  $(a, b)$  and crossing the center of the ring, then  $\sum_\nu \cos 2k_n x_\nu = 0$  and the amplitude of the dip in  $\mathcal{T}(\phi)$  goes to zero with increasing  $r$  for  $\sqrt{N}|r| \gg \gamma$ . Finally, if impurities are distributed symmetrically with respect to the ring center, one gets  $\sum_\nu \exp(2ik_n x_\nu) = 0$  and dependence  $\mathcal{T}(\phi)$  becomes the same as in the clean ring.

## III. THE RING WITH IMPURITIES AND SPIN-ORBIT INTERACTION

In this section we discuss the effect of impurities on the conductance of a ring with spin-orbit and Zeeman interactions. The case of a clean ring was studied in detail in Ref. 52.

The Hamiltonian of a clean ring with SO-interaction induced by axially-symmetric built-in field is given by

$$\hat{H} = \hat{H}_{kin} + \hat{H}_Z + \hat{H}_{SO}, \quad (50)$$

where

$$\hat{H}_{kin} = -\frac{\hbar^2}{2m} D_x^2, \quad (51)$$

is the kinetic energy,  $D_x = \partial/\partial x + 2\pi i\phi/L$ ,

$$\hat{H}_Z = \frac{1}{2}\hbar\omega_Z \hat{\sigma}_z, \quad (52)$$

is the Zeeman term ( $\hbar\omega_Z$  is the Zeeman splitting energy in the external magnetic field parallel to the  $z$  axis) and  $\hat{H}_{SO}$  describes the SO coupling:

$$\hat{H}_{SO} = -i\xi \frac{\hbar^2}{2m} \left\{ \begin{bmatrix} -\cos\theta & \sin\theta e^{-2\pi i x/L} \\ \sin\theta e^{2\pi i x/L} & \cos\theta \end{bmatrix}, D_x \right\}. \quad (53)$$

Here  $\theta$  is the angle between effective SO-induced magnetic field and the  $z$  axis,  $\xi$  is the dimensionless parameter characterizing the strength of SO interaction,  $\{\dots\}$  stands for the anticommutator.

The problem is studied in the quasiclassical case ( $k_F L \gg 1, \xi \ll k_F L$ ) in which the effect of the SO interaction is described by the rotation of the electron spin



in effective magnetic field which varies along the electron trajectory. The stationary wavefunctions and energies in this case read:

$$\begin{aligned}\psi_{n,\pm}^{(1)}(x) &= e^{\pm i2\pi nx/L} \begin{bmatrix} \cos \vartheta_{\pm}/2 \\ -\sin \vartheta_{\pm}/2 e^{2\pi ix/L} \end{bmatrix}, \\ \psi_{n,\pm}^{(2)}(x) &= e^{\pm i2\pi nx/L} \begin{bmatrix} \sin \vartheta_{\pm}/2 e^{-2\pi ix/L} \\ \cos \vartheta_{\pm}/2 \end{bmatrix}, \\ \epsilon_{n,+}^{(1)} &= \Delta(n + \phi - \delta_+), \quad \epsilon_{n,-}^{(1)} = \Delta(n - \phi + \delta_-), \\ \epsilon_{n,+}^{(2)} &= \Delta(n + \phi + \delta_+), \quad \epsilon_{n,-}^{(2)} = \Delta(n - \phi - \delta_-).\end{aligned}\quad (54)$$

Here we introduced the notations

$$\begin{aligned}\delta_{\pm} &= |\kappa_{\pm}| - \frac{1}{2}, \quad e^{i\vartheta_{\pm}} = \frac{\kappa_{\pm}}{|\kappa_{\pm}|}, \\ \kappa_{\pm} &= \frac{1}{2} + \xi e^{i\theta} \mp \Omega_Z, \\ \Omega_Z &= \omega_Z L / 4\pi v_F.\end{aligned}\quad (55)$$

As seen from Eq.(54) the degeneracy of the levels occurs for the following eight values of magnetic flux:  $\phi = \pm\delta, \phi = \pm\delta', \phi = 1/2 \pm \delta$  and  $\phi = 1/2 \pm \delta'$ , where  $\delta = (\delta_+ + \delta_-)/2, \delta' = (\delta_+ - \delta_-)/2 + 1/2$ . However, at four of these values,  $\phi = \pm\delta$  and  $\phi = \pm\delta'$  the resonances are absent because of the backscattering on the contacts (see appendix B in Ref. 52). At four other points, the backscattering is negligible, and there appear the antiresonances with the width  $\gamma$ . The amplitudes of the antiresonances at  $\phi = 1/2 \pm \delta$  and  $\phi = 1/2 \pm \delta'$  are  $\gamma c^2$  and  $\gamma s^2$ , respectively, where  $c = \cos(\vartheta_+ - \vartheta_-)/2, s = \sin(\vartheta_+ - \vartheta_-)/2$ .

At  $\phi = 1/2 + \delta$ , the degeneracy occurs between the states  $\psi_{n,+}^{(1)}$  and  $\psi_{n+1,-}^{(1)}$ ; at  $\phi = 1/2 + \delta$ , between  $\psi_{n,+}^{(1)}$  and  $\psi_{n+1,-}^{(2)}$ ; at  $\phi = \delta' - 1/2$ , between  $\psi_{n,+}^{(1)}$  and  $\psi_{n,-}^{(2)}$ ; and, finally, at  $\phi = -\delta' + 1/2$ , between  $\psi_{n,+}^{(2)}$  and  $\psi_{n,-}^{(1)}$ . We note that the amplitudes of the antiresonances ( $\gamma c^2$  and  $\gamma s^2$ ) are determined by the scalar products of the corresponding spinors:  $c^2 = |\langle \psi_{n,+}^{(1)} | \psi_{n+1,-}^{(1)} \rangle|^2 = |\langle \psi_{n,+}^{(2)} | \psi_{n+1,-}^{(2)} \rangle|^2$  and  $s^2 = |\langle \psi_{n,+}^{(2)} | \psi_{n,-}^{(1)} \rangle|^2 = |\langle \psi_{n,+}^{(1)} | \psi_{n,-}^{(2)} \rangle|^2$ .

The relation between the transmission coefficient and the stationary states of the closed ring, derived in Appendix B, allows us to easily find out the influence of the impurities on the four resonances, described above. As in the previous section, we use the two-level approximation (we assume that the distance between the antiresonances is much larger than  $\sqrt{N}|r|$ ). The matrix elements of impurity potential read:

$$\begin{aligned}\langle \psi_{n,+}^{(1)} | \hat{V} | \psi_{m,-}^{(1)} \rangle &= \langle \psi_{n,+}^{(2)} | \hat{V} | \psi_{m,-}^{(2)} \rangle = cV_{nm}, \\ \langle \psi_{n,+}^{(2)} | \hat{V} | \psi_{m,-}^{(1)} \rangle &= -\langle \psi_{n,+}^{(1)} | \hat{V} | \psi_{m,-}^{(2)} \rangle = sV_{nm},\end{aligned}\quad (56)$$

where  $V_{nm} = i\Delta r \sum_{\nu} \exp(i(n+m)2\pi x_{\nu}/L)/2\pi$  is the matrix element, appearing in the spinless problem.

Using Eqs. (54), (56) and (14) (the latter equation was modified for the spinful case) we obtain the following result for the transmission coefficient:

$$\begin{aligned}T_{SO}(\phi) &= \frac{c^2}{2} [\mathcal{T}(\phi + \delta; rc) + \mathcal{T}(\phi - \delta; rc)] \\ &+ \frac{s^2}{2} [\mathcal{T}(\phi + \delta'; rs) + \mathcal{T}(\phi - \delta'; rs)],\end{aligned}\quad (57)$$

where  $\mathcal{T}(\phi; \rho)$  is the transmission coefficient in the spinless problem (given by Eq.(34)) with the reflection amplitude  $r$  substituted by  $\rho$ . We see that the effect of the impurity scattering in the spinless and in the spinful case is essentially the same: the antiresonances are broadened and their amplitudes become smaller (for strong enough impurities the amplitudes are two times smaller than in a clean ring). The only difference is the appearance of "effective" reflection amplitudes  $rc$  and  $rs$  for the antiresonances at  $\phi = 1/2 \pm \delta$  and  $\phi = 1/2 \pm \delta'$ , respectively.

The expression for transmission coefficient is especially simple in the absence of Zeeman interaction (in this case  $s = 0, c = 1$ ):

$$T_{SO}(\phi) = \frac{1}{2} [\mathcal{T}(\phi + \delta; r) + \mathcal{T}(\phi - \delta; r)].\quad (58)$$

This equation shows two types of periodic oscillations: AB oscillations with magnetic flux and AC oscillations with  $\delta$ .

The obtained results are illustrated in Fig. 7.

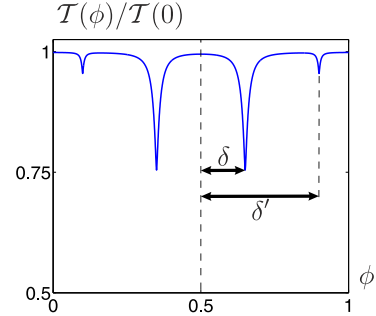


FIG. 7: Transmission coefficient in the ring with impurities. Both spin-orbit and Zeeman interactions are present.

#### IV. CONCLUSION

In this work, we have studied the effect of the disorder on the Aharonov-Bohm interferometer made of a single-channel non-interacting quantum ring tunnel-coupled to the leads. We focused on the case of large temperature (compared to the level spacing) and weak tunneling coupling. In this case tunneling conductance exhibits sharp dips at half-integer values of the magnetic flux. We demonstrated that the short-range potential broadens these dips, while the long-range smooth disorder leads to appearing of negative resonant peaks at integer values of the flux. We also found analytical expression for the shape of the peaks which turned out to be essentially non-Lorentzian. The results have been generalized to account

for the spin-orbit interaction which leads to splitting of the disorder-broadened resonant peaks, and for the Zeeman coupling which results in arising of additional peaks in the tunneling conductance.

## V. ACKNOWLEDGMENTS

We thank I.V. Gornyi and D.G. Polyakov for fruitful discussions and useful comments. The work was supported by RFBR, by programmes of the RAS, by RF President Grant NSH-5442.2012.2, and by the Dynasty foundation.

## Appendix A

In this appendix, we calculate the sum

$$\sum_{n=0}^{\infty} \left| \left( \boldsymbol{\alpha}, \hat{A}^n \boldsymbol{\beta} \right) \right|^2, \quad (\text{A1})$$

where  $\hat{A}$  is an arbitrary  $2 \times 2$  matrix, and  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are arbitrary two-component vectors. First, we note that

$$\hat{A}^n = \Sigma_n + \Delta_n (\hat{A} - \text{Tr} \hat{A} / 2), \quad (\text{A2})$$

where  $\Sigma_n = (\lambda_1^n + \lambda_2^n)/2$ ,  $\Delta_n = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$ . To prove Eq. (A2), one can make a similarity transformation that reduces  $\hat{A}$  to a diagonal (or Jordan) form. Using Eq. (A2) we can rewrite Eq. (A1) as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \left( \boldsymbol{\alpha}, \hat{A}^n \boldsymbol{\beta} \right) \right|^2 = & \\ & |a|^2 \sum_{n=0}^{\infty} |\Sigma_n|^2 + |b - a \text{Tr} \hat{A} / 2|^2 \sum_{n=0}^{\infty} |\Delta_n|^2 \\ & + 2\text{Re}[a^* (b - a \text{Tr} \hat{A} / 2) \sum_{n=0}^{\infty} \Sigma_n^* \Delta_n], \end{aligned} \quad (\text{A3})$$

where  $a = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ ,  $b = (\boldsymbol{\alpha}, \hat{A} \boldsymbol{\beta})$ . The sums entering Eq. (A3) are easily calculated:

$$\begin{aligned} \sum_{n=0}^{\infty} |\Sigma_n|^2 &= \frac{1}{4} \left( \frac{1}{1 - |\lambda_1|^2} + \frac{1}{1 - |\lambda_2|^2} + 2\text{Re} \frac{1}{1 - \lambda_1^* \lambda_2} \right), \\ \sum_{n=0}^{\infty} |\Delta_n|^2 &= \frac{1}{|\lambda_1 - \lambda_2|^2} \left( \frac{1}{1 - |\lambda_1|^2} + \frac{1}{1 - |\lambda_2|^2} - 2\text{Re} \frac{1}{1 - \lambda_1^* \lambda_2} \right), \\ \sum_{n=0}^{\infty} \Sigma_n^* \Delta_n &= \frac{1}{2(\lambda_1 - \lambda_2)} \left( \frac{1}{1 - |\lambda_1|^2} - \frac{1}{1 - |\lambda_2|^2} + 2i\text{Im} \frac{1}{1 - \lambda_1^* \lambda_2} \right) \end{aligned}$$

These expressions can be rewritten in terms of  $D = \det \hat{A}$ ,  $S = \text{Tr} \hat{A}$ . After some algebra we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} |\Sigma_n|^2 &= \frac{1}{4} Z^{-1} [4(1 - |D|^2) - |S|^2(|D|^2 + 3) + 2\text{Re} D^* S^2] \\ \sum_{n=0}^{\infty} |\Delta_n|^2 &= Z^{-1} (1 - |D|^2), \\ \sum_{n=0}^{\infty} \Sigma_n^* \Delta_n &= \frac{1}{2} Z^{-1} [S^* (1 + |D|^2) - 2D^* S], \end{aligned} \quad (\text{A4})$$

where  $Z = (1 - |D|^2)^2 - (1 + |D|^2)|S|^2 + 2\text{Re} D^* S^2$ . Finally, we get

$$\sum_{n=0}^{\infty} \left| \left( \boldsymbol{\alpha}, \hat{A}^n \boldsymbol{\beta} \right) \right|^2 = \frac{|a|^2(1 - |D|^2 - |S|^2 - |DS|^2 + 2\text{Re} D^* S^2) + |b|^2(1 - |D|^2) + 2\text{Re}[a^* b(S^* |D|^2 - D^* S)]}{(1 - |D|^2)^2 - (1 + |D|^2)|S|^2 + 2\text{Re} D^* S^2}. \quad (\text{A5})$$

In the case of real  $D$  and  $S$  this expression is simplified:

$$\sum_{n=0}^{\infty} \left| \left( \boldsymbol{\alpha}, \hat{A}^n \boldsymbol{\beta} \right) \right|^2 = \frac{|a|^2(1 + D - S^2 + DS^2) + |b|^2(1 + D) - 2DS\text{Re}(a^* b)}{(1 - D)[(1 + D)^2 - S^2]}. \quad (\text{A6})$$

## Appendix B

In this Appendix we show that the transmission coefficient may be easily expressed in terms of stationary

levels of the closed ring. We start from discussion of the clean ring and then generalize obtained results for the ring with short-range disorder.

## 1. Clean ring

Using Eqs. (2), (6) and (10), one can write the transmission coefficient in the following way

$$\mathcal{T}(\phi) = \left\langle \left| \left( \mathbf{e}, \frac{1}{1 - e^{ikL\hat{A}}} \beta_0 \right) \right|^2 \right\rangle_E, \quad (\text{B1})$$

Neglecting the backscattering by the contacts [see equation (28)] we get

$$\mathcal{T}(\phi) = t^2 t_{out}^2 \left\langle \left| \frac{e^{-i\pi\phi}}{1 - (1 - 2\gamma)e^{i(kL - 2\pi\phi)}} + \frac{e^{i\pi\phi}}{1 - (1 - 2\gamma)e^{i(kL + 2\pi\phi)}} \right|^2 \right\rangle_E, \quad (\text{B2})$$

For  $\gamma = 0$ , the integrand of Eq. (B1) has poles on the real axis,  $k_n^\pm = 2\pi(n \pm \phi)/L$ . These poles are related to the energy levels of the closed ring in following way [see Eq. (18)]:  $\epsilon_n^\pm = \hbar v_F k_n^\pm + c$  (it is notable that  $k_n^\pm$  do not coincide with the eigenvalues of momentum operator). Expanding denominators of these fractions near  $k_n^\pm$ , after simple algebra we find that Eq. (B2) approximately coincides with Eq. (14) of the main text where  $G_E$  is found from Eq. (15). In this case, the wave functions entering Eq. (15) are simply given by  $\psi_n^\pm(0) = 1/\sqrt{L}$  and  $\psi_n^\pm(L/2) = (-1)^n/\sqrt{L}$ . Below we demonstrate that Eq. (14) also holds for dirty ring where wave functions strongly depend on realization of disorder.

## 2. Ring with short-range disorder

The transmission coefficient of disordered ring is also given by Eq. (B1) where both  $\hat{A}$  and  $\beta_0$  depend on disorder. As we discussed in Sec. IIB the matrix  $\hat{A}$  can be approximately written as  $\hat{A} = (1 - 2\gamma)\hat{A}_0$ , where the unitary matrix  $\hat{A}_0$  is given by Eq. (31). Let us also introduce matrix  $\hat{U}$ , such that  $e^{ikL/2}\hat{U}$  is a transfer matrix from contact  $a$  to contact  $b$ . Then, one can express  $\beta_0$  in terms of this matrix  $\beta_0 = e^{ikL/2} t t_{out} \hat{U} \mathbf{e}$ . Denoting eigenvalues of  $e^{ikL} \hat{A}_0$  as  $e^{iQ_1(k)L}$  and  $e^{iQ_2(k)L}$  and corresponding eigenvectors as  $\chi_1(k)$  and  $\chi_2(k)$ , we rewrite Eq. (B1) as follows

$$\mathcal{T}(\phi) = t^2 t_{out}^2 \left\langle \left| \sum_{\alpha=1,2} \frac{(\mathbf{e}, \chi_\alpha)(\chi_\alpha, \hat{U} \mathbf{e})}{1 - (1 - 2\gamma)e^{iQ_\alpha(k)L}} \right|^2 \right\rangle_E. \quad (\text{B3})$$

Equation (B3) is a generalization of Eq. (B2) for a disordered ring. [In the clean ring,  $Q_1(k) = k - 2\pi\phi/L$ ,  $Q_2(k) = k + 2\pi\phi/L$ ,  $\chi_1 = (1, 0)$ ,  $\chi_2 = (0, 1)$ , and  $\hat{U}$  is diagonal matrix with the elements  $\exp(-i\pi\phi)$  and  $\exp(i\pi\phi)$ .]

As was pointed out in the section IIB, the matrix  $\hat{A}$  relates the amplitudes  $C_\pm, D_\pm$  of the waves in the vicinity of the contact  $b$  to each other [see Fig. 5 and Eq. (30)]. In the closed ring ( $\gamma = 0$ ) the stationary states can be found from the conditions  $C_+ = D_+$  and  $C_- = D_-$ . This allows us to establish a relation between the matrix  $\hat{A}_0$  and the stationary states of the closed ring. Specifically, the wavevectors  $k = k_{1n}$  and  $k = k_{2n}$ , for which one of the eigenvalues of the matrix  $e^{ikL} \hat{A}_0$  equals to unity, correspond to the energy levels. These wavevectors are found from the equation:

$$Q_\alpha(k_{\alpha n}) = 2\pi n/L$$

Each of the corresponding eigenvectors with components  $(C_+, C_-)$  describes the stationary wavefunction in the vicinity of the point  $b$  ( $x$  close to  $L/2$ ):

$$\psi(x) = \frac{C_+ e^{i(k_{\alpha n} - 2\pi\phi/L)(x - L/2)} + C_- e^{-i(k_{\alpha n} + 2\pi\phi/L)(x - L/2)}}{\sqrt{L}}.$$

As expected, for  $\gamma = 0$  Eq. (B3) has poles as a function of  $k$  for  $k = k_{1n}$  and  $k = k_{2n}$ .

For almost closed ring the poles of the r.h.s. of Eq. (B3) slightly shift away from the axis of real  $k$ . Just as in the clean ring, the main contribution to the integral over  $E$  comes from the poles of the fractions  $[1 - (1 - 2\gamma)e^{iQ_\alpha(k)L}]^{-1}$ , while the terms  $(\mathbf{e}, \chi_\alpha)(\chi_\alpha, \hat{U} \mathbf{e})$  as well as function  $\partial f / \partial E$  can be taken at the poles for  $\gamma = 0$ . Next, we notice that the vectors  $\chi_\alpha$  and the matrix  $\hat{U}$  are defined in such a way that

$$\begin{aligned} (\mathbf{e}, \chi_\alpha)|_{k=k_{\alpha n}} &= \sqrt{L} \psi_{\alpha n}(L/2), \\ (\chi_\alpha, \hat{U} \mathbf{e})|_{k=k_{\alpha n}} &= \sqrt{L} \psi_{\alpha n}^*(0), \end{aligned}$$

where  $\psi_{\alpha n}(x)$  are the stationary wave functions of the ring with disorder. Using these equations, expanding denominators in Eq. (B3) near the poles and neglecting  $dQ_\alpha/dk - 1 \sim \sqrt{N}|r| \ll 1$  with respect to unity, we arrive to Eqs. (14) and (15).

<sup>1</sup> Y. Aharonov, D. Bohm, Phys. Rev. B **115**, 485 (1959).

<sup>2</sup> A.G. Aronov and Yu.V. Sharvin, Rev. Mod. Phys. **59**, 755 (1987).

<sup>3</sup> A. Yacoby, M. Heiblum, D. Mahalu, and H. Shtrikman, Phys. Rev. Lett. **74**, 4047 (1995).

<sup>4</sup> A. Yacoby, R. Schuster, and M. Heiblum Phys. Rev. B **53**, 9583 (1996)

<sup>5</sup> A. van Oudenaarden, M. H. Devoret, Yu.V. Nazarov, and J. E. Mooij, Nature **391**, 768 (1998).

<sup>6</sup> A. A. Bykov, A. K. Bakarov, L. V. Litvin, and A. I.

- Toropov, JETP Letters **72**, 209 (2000).
- <sup>7</sup> A. A. Bykov, D. G. Baksheev, L. V. Litvin, V. P. Migal, E. B. Olshanetskii, M. Casse', D. K. Maude, and J. C. Portal, JETP Letters, **71**, 434 (2000).
  - <sup>8</sup> O. M. Auslaender, A. Yacoby, R. de Picciotto, K. W. Baldwin, L. N. Pfeiffer, and West K W, Science **295**, 825 (2002).
  - <sup>9</sup> Yang Ji, Yunchul Chung, D. Sprinzak, M. Heiblum, D. Mahalu, Hadas Shtrikman, Nature **422**, 415 (2003)
  - <sup>10</sup> P. Samuelsson, E. V. Sukhorukov, M. Buttiker, Phys. Rev. Lett. **92**, 026805 (2004).
  - <sup>11</sup> M. Avinun-Kalish, M. Heiblum, O. Zarchin, D. Mahalu, V. Umansky, Nature **436**, 529 (2005).
  - <sup>12</sup> I. Neder, M. Heiblum, Y. Levinson, D. Mahalu, and V. Umansky, Phys. Rev. Lett. **96**, 016804 (2006).
  - <sup>13</sup> I. Neder, N. Ofek, Y. Chung, M. Heiblum, D. Mahalu, V. Umansky, Nature **448**, 333 (2007).
  - <sup>14</sup> I. Neder, M. Heiblum, D. Mahalu, and V. Umansky, Phys. Rev. Lett. **98**, 036803 (2007)
  - <sup>15</sup> Preden Roulleau, F. Portier, D. C. Glatthli, and P. Roche, A. Cavanna, G. Faini, U. Gennser, and D. Mailly, Phys Rev B **76**, 161309 (2007).
  - <sup>16</sup> Preden Roulleau, F. Portier, and P. Roche Phys. Rev. Lett. **100**, 126802 (2008).
  - <sup>17</sup> Dong-In Chang, Gyong Luck Khym, Kicheon Kang, Yunchul Chung, Hu-Jong Lee, Minky Seo, Moty Heiblum, Diana Mahalu, Vladimir Umansky Nature Physics **4**, 205 (2008).
  - <sup>18</sup> Yiming Zhang, D. T. McClure, E. M. Levenson-Falk, and C. M. Marcus, L. N. Pfeiffer and K. W. West, Phys. Rev. B **79**, 241304 (2009).
  - <sup>19</sup> N. Ofek, Aveek Bid, M. Heiblum, Ady Stern, V. Umansky, and D. Mahalu PNAS **107**, 5276 (2010).
  - <sup>20</sup> E. Weisz, H. K. Choi, M. Heiblum, Yuval Gefen, V. Umansky, and D. Mahalu Phys. Rev. Lett. **109**, 250401 (2012).
  - <sup>21</sup> M. Büttiker, Y. Imry, and M.Ya. Azbel, Phys. Rev. A **30**, 1982 (1984); Y. Gefen, Y. Imry, and M.Ya. Azbel, Phys. Rev. Lett. **52**, 129 (1984); M. Büttiker, Y. Imry, R. Landauer, S. Pinhas, Phys. Rev. B **31**, 6207 (1985).
  - <sup>22</sup> M.V. Moskalets, Low Temp. Phys. **23**, 824 (1997).
  - <sup>23</sup> Qiming Li and C. M. Soukoulis, Phys. Rev. B **33**, 7318 (1986).
  - <sup>24</sup> J. M. Mao, Y. Huang, and J. M. Zhou, J. Appl. Phys. **73**, 1853 (1993).
  - <sup>25</sup> E.P. Nakhmedov, H.Feldmann, and R. Oppermann, Eur. Phys. J. B **16**, 515 (2000).
  - <sup>26</sup> M.A. Kokoreva, V.A. Margulis, M.A. Pyataev, Physica E, **43**, 1610 (2011).
  - <sup>27</sup> J.M. Kinaret, M. Jonson, R.I. Shekhter, S. Eggert, Phys. Rev. B **57**, 3777 (1998).
  - <sup>28</sup> M. Eroms, L. Mayrhofer, and M. Grifoni, Phys. Rev. B **78** 075403 (2008).
  - <sup>29</sup> E.A. Jagla, C.A. Balseiro, Phys. Rev. Lett. **70**, 639 (1993).
  - <sup>30</sup> A.P. Dmitriev, I.V. Gornyi, V.Yu. Kachorovskii, D.G. Polyakov Phys. Rev. Lett., **105**, 036402 (2010).
  - <sup>31</sup> Y. Aharonov, A. Casher, Phys. Rev. Lett **53**, 319 (1984)
  - <sup>32</sup> H. Mathur, A.D. Stone, Phys. Rev. Lett **68**, 2964 (1992)
  - <sup>33</sup> H. Mathur, A.D. Stone, Phys. Rev. B **44**, 10957 (1991).
  - <sup>34</sup> A.G. Aronov, Y.B. Lyanda-Geller, Phys. Rev. Lett **70**, 343 (1993).
  - <sup>35</sup> T.Z. Qian, Z.B. Su, Phys. Rev. Lett. **72**, 2311 (1994).
  - <sup>36</sup> J. Nitta, F.E. Meijer, H. Takayanji, Appl. Phys. Lett **75**, 695 (1999).
  - <sup>37</sup> D. Frustaglia, K. Richter, Phys. Rev. B **69**, 235310 (2004).
  - <sup>38</sup> B. Molnar, F.M. Peeters, P. Vasilopoulos, Phys. Rev. B **69**, 155335 (2004).
  - <sup>39</sup> U. Aeberhard, K. Wakabayashi, M. Sigrist, Phys. Rev. B **72**, 075328 (2005).
  - <sup>40</sup> R. Citro, F. Romeo, Phys. Rev. B **73**, 233304 (2006).
  - <sup>41</sup> M. Pletyukhov, V. Gritsev, N. Pauget, Phys. Rev. B **74**, 045301 (2006).
  - <sup>42</sup> R. Citro, F. Romeo, M. Marinaro, Phys. Rev. B **74**, 115329 (2006).
  - <sup>43</sup> A.A. Kovalev, M.F. Borunda, T. Jungwirth, L.W. Molenkamp, J. Sinova, Phys. Rev. B **76**, 125307 (2007).
  - <sup>44</sup> F. Romeo, R. Citro, M. Marinaro, Phys. Rev. B **78**, 245309 (2008).
  - <sup>45</sup> A.M. Lobos and A.A. Aligia, Phys. Rev. Lett **100**, 016803 (2008).
  - <sup>46</sup> M. Pletyukhov and U. Zülicke, Phys. Rev. B **77**, 193304 (2008).
  - <sup>47</sup> V. Moldoveanu and B. Tanatar, Phys. Rev. B **81**, 035326 (2010).
  - <sup>48</sup> A. Aharony, Y. Tokura, G.Z. Cohen, O. Entin-Wohlman, S. Katsumoto, Phys. Rev. B **84**, 035323 (2011).
  - <sup>49</sup> P. Michetti and P. Recher, Phys. Rev. B **83**, 125420 (2011).
  - <sup>50</sup> M. König, A. Tschetschetkin, E.M. Hankiewicz, J. Sinova, V. Hock, V. Daumer, M. Schafer, C.R. Becker, H. Buhmann, L.W. Molenkamp, Phys. Rev. Lett. **96**, 076804 (2006).
  - <sup>51</sup> T. Bergsten, T. Kobayashi, Y. Sekine, J. Nitta, Phys. Rev. Lett. **97**, 196803 (2006).
  - <sup>52</sup> P. M. Shmakov, A. P. Dmitriev, and V. Yu. Kachorovskii Phys. Rev. B **85**, 075422 (2012)
  - <sup>53</sup> The AB conductance through the single-channel ring with a scattering potential barrier in one of the arms was discussed in Ref. 22 for the case of almost transparent contacts ( $\gamma \simeq 1$ ). It was shown that interference part of the conductance is not entirely suppressed for  $T \gg \Delta$ .
  - <sup>54</sup> In this case the localization effects are absent.